MATH Math 162A Review: Inner Product

1. Let x, y be non-zero vectors. Prove that

$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\| = \frac{\|x-y\|}{\|x\| \cdot \|y\|}.$$

Solution:

$$\left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\|^2 = \left\langle \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2}, \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\rangle$$

$$= \frac{\langle x, x\rangle}{\|x\|^4} + \frac{\langle y, y\rangle}{\|y\|^4} - 2\frac{\langle x, y\rangle}{\|x\|^2\|y\|^2}$$

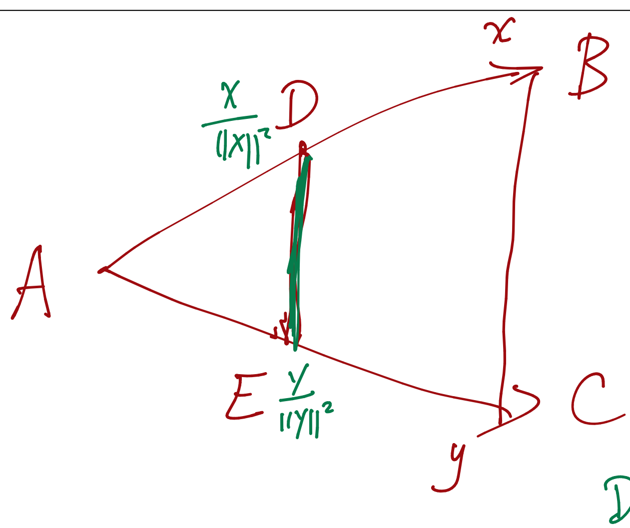
$$= \frac{1}{\|x\|^2} + \frac{1}{\|y\|^2} - 2\frac{\langle x, y\rangle}{\|x\|^2\|y\|^2}$$

$$= \frac{1}{\|x\|^2\|y\|^2}\left(\|y\|^2 + \|x\|^2 - 2\langle x, y\rangle\right)$$

$$= \frac{\|x-y\|^2}{\|x\|^2 \cdot \|y\|^2}$$

By the way, $x \mapsto \frac{x}{\|x\|^2}$ is called Kelvin Transformation

Here is a geometric interpretion of the transformat



$\triangle ABC$ is similar to $\triangle AED$. Thus

$$\frac{DE}{BC} = \frac{AD}{AC}$$

$$DE = \frac{BC \cdot AD}{AC} = \frac{\|x-y\|}{\|x\| \cdot \|y\|}$$

2. Prove the Cauchy-Schwarz inequality of the following form

$$\left|\int_0^1 f(x)g(x)dx\right|^2 \leq \int_0^1 f(x)^2dx \cdot \int_0^1 g(x)^2dx.$$

(historic note: should be more fair to call it "Cauchy-Bunyakovsky-Schwarz inequality". Cauchy proved the version for sums, and Bunyakovsky proved the above version many years prior to Schwarz.)

Solution:

We just need to prove that

$$(f,g) = \int_0^1 f(x)g(x)\,dx$$

is an inner product, and thus the inequality follows from the general Cauchy inequality.

① Symmetry $\int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx$

Obvious.

② Linearity $\int_0^1 f(x)(a\,g(x)+b\,h(x))dx$

$$= a\int_0^1 f(x)g(x)dx + b\int_0^1 f(x)h(x)dx$$

✓

③ positivity $\int_0^1 f(x)^2dx \geq 0$, and if $\int_0^1 f(x)^2dx = 0$. Assume $f(x)$ is continuous. Then $f(x) \equiv 0$. otherwise if $f(x_0) \neq 0$. by continuity, $\exists\ \delta$ such that $|x-x_0| \leq \delta \Rightarrow |f(x)-f(x_0)| < \frac{1}{2}|f(x_0)| \Rightarrow |f(x)| > \frac{1}{2}|f(x_0)|$ Thus

$$0 = \int_0^1 f(x)^2dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)^2dx \geq \frac{1}{4}|f(x_0)|^2 \cdot 2\delta \neq 0.$$